Optimality of Intermediate-Thrust Arcs of Rocket Trajectories

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A new necessary condition is derived for the optimality of intermediate-thrust arcs of rocket trajectories in vacuum. The new necessary condition, together with previously known necessary conditions, ensures the absolute optimality of "sufficiently short" intermediate-thrust arcs. For inverse-square central fields, the new condition reduces to a requirement that the primer vector must point inward; its radial component must be negative. Intermediate-thrust arcs that satisfy the new condition cannot join to arcs of a finite maximum thrust, nor join directly (without impulses) to arcs of zero thrust, except in a very restricted set of special cases. They are, therefore, of theoretical rather than practical importance.

1. Introduction

IT is well known from the calculus of variations that optimal rocket trajectories comprise three types of arc: zero-thrust coasting arcs, maximum-thrust arcs (that reduce to impulses if thrust is unbounded), and intermediate-thrust arcs (i.e., arcs in which the thrust is greater than zero but less than the maximum possible thrust). An intermediate-thrust arc constitutes a singular case, in which the thrust level is not determined directly from the first-order variational equations, but in a more subtle indirect manner. For essentially one-dimensional problems of rocket flight in atmosphere, such arcs are known to play an important role, and their optimality can be established by the Green's theorem method of Miele.[1] For two- and three-dimensional trajectories in vacuum, which are the sole concern of the present paper, the situation is less satisfactory. Intermediate-thrust extremals are known to exist; Lawden[2,3] has exhibited explicit solutions for the two-dimensional case with an inverse-square field. However, there has been uncertainty as to when, if ever, intermediate-thrust arcs are truly optimal rather than merely stationary, and uncertainty as to when, if ever, they need be considered for the solution of practical problems.

These uncertainties are partially resolved by the results of the present paper. Section 2 summarizes the previously known necessary conditions for intermediate-thrust trajectories in vacuum and shows that intermediate-thrust arcs divide naturally into two classes, according to the sign of a certain quantity q. Section 3 shows that $q \geq 0$ is a necessary condition for optimality. Section 4 shows that a "sufficiently short" intermediate-thrust arc with fixed end points is an absolute optimum if it satisfies all the previously known necessary conditions and also satisfies the new condition in its ($q > 0$). Section 5 considers the problem of joining intermediate thrust arcs with $q > 0$ to zero-thrust arcs and maximum-thrust arcs to make an optimal trajectory satisfying given end conditions. It concludes that this is impossible (except in very restrictive special cases) if the allowable thrust has a finite upper limit, however large. Section 6 discusses the theoretical and practical implications of the results.

2. Summary of Previously Known Necessary Conditions

It has been shown by many authors that an optimal rocket trajectory in vacuum must satisfy the equations

$$\ddot{\mathbf{r}} = \mathbf{g}(\mathbf{r}, t) + \mathbf{a} \tag{1}$$

$$\mathbf{a} = a\boldsymbol{\lambda}/\lambda \tag{2}$$

$$\ddot{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}\cdot\nabla)\mathbf{g} \tag{3}$$

where $\mathbf{r}$ is the rocket's position vector, $\mathbf{g} = -\nabla U$ is the gravity vector derived from a potential U, and $\boldsymbol{\lambda}$ is the "primer vector" of Lawden.[2] If the rocket has a maximum thrust F, mass m, and effective exhaust velocity c, the magnitude of thrust acceleration is governed by the further equations

$$a = SF/m \tag{4}$$

$$\dot{m} = -SF/c \tag{5}$$

$$\dot{K} = \dot{\lambda}/m \tag{6}$$

where S is a throttle variable with range 0 to 1, and K is a "switch function" that determines the optimal value of S according to the rule

$$S = 0 \qquad \text{if} \qquad K < 0 \tag{7a}$$

$$S = 1 \qquad \text{if} \qquad K > 0 \tag{7b}$$

$$0 \leq S \leq 1 \qquad \text{if} \qquad K = 0 \tag{7c}$$

Equations (7a–7c) correspond to coast arcs, maximum-thrust arcs, and intermediate-thrust arcs, respectively. During intermediate-thrust arcs, $K \equiv 0$, which, by Eq. (6) implies a constant magnitude for the primer vector. That is

$$\lambda^2 \equiv \boldsymbol{\lambda}\cdot\boldsymbol{\lambda} = \text{const} \tag{8}$$

during intermediate-thrust arcs. On going from an intermediate-thrust arc to an adjoining arc, K must decrease or increase, accordingly, as the adjoining arc is of zero or maximum thrust. The same behavior must be shown by λ, since $\dot{K}$ and $\dot{\lambda}$ always agree in sign [Eq. (6)]. If there is no upper bound on thrust magnitude, the maximum-thrust arcs reduce to impulses, and Eqs. (4–7) reduce to the single condition

$$a(t) = 0 \qquad \text{if} \qquad \lambda(t) < \max(\lambda) \tag{9}$$

That is, thrust acceleration occurs only when λ attains its maximum value. If the maximum is only attained for an instant, a thrust impulse occurs. If the maximum is attained for a finite period, an intermediate-thrust arc occurs. This arc may have impulses at each end, but λ must decrease beyond the ends of the intermediate-thrust arc, whether the impulses are present or not.

Repeated differentiation of the identity $\lambda^2 \equiv \boldsymbol{\lambda}\cdot\boldsymbol{\lambda}$ and use of Eqs. (1–3) gives the identities

$$\lambda\dot{\lambda} = \boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}} \tag{10}$$

$$\dot{\lambda}\dot{\lambda} + \lambda\ddot{\lambda} = \dot{\boldsymbol{\lambda}}\cdot\dot{\boldsymbol{\lambda}} - (\boldsymbol{\lambda}\cdot\nabla)^2U \tag{11}$$

$$3\dot{\lambda}\ddot{\lambda} + \lambda\dddot{\lambda} = -(\dot{\mathbf{r}}\cdot\nabla)(\boldsymbol{\lambda}\cdot\nabla)^2U - 4(\boldsymbol{\lambda}\cdot\nabla)(\dot{\boldsymbol{\lambda}}\cdot\nabla)U - (\boldsymbol{\lambda}\cdot\nabla)^2\partial U/\partial t \tag{12}$$

and

$$3\ddot{\lambda}\ddot{\lambda} + 4\dot{\lambda}\dddot{\lambda} + \lambda\lambda^{(4)} = -a(\boldsymbol{\lambda}\cdot\nabla)^3U + W(\boldsymbol{\lambda},\dot{\boldsymbol{\lambda}}, \mathbf{r}, \dot{\mathbf{r}}, t) \tag{13}$$

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where W is a fairly complicated expression, whose form is not important for present purposes. For an intermediate-thrust arc, the constancy of λ makes the left sides of Eqs. (10–13) vanish, giving

$$0 = \boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}} \tag{10'}$$

$$0 = \dot{\boldsymbol{\lambda}}\cdot\dot{\boldsymbol{\lambda}} - (\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})^2 U \tag{11'}$$

$$0 = -(\dot{\mathbf{r}}\cdot\boldsymbol{\nabla})(\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})^2 U - 4(\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})(\dot{\boldsymbol{\lambda}}\cdot\boldsymbol{\nabla})U - (\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})^2\,\partial u/\partial t \tag{12'}$$

$$0 = -aq + W \tag{13'}$$

where use has been made of the abbreviation

$$q = (\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})^3 U \tag{14}$$

It is evident that if Eqs. (10′–12′) hold simultaneously at any instant, they will continue to hold for as long as the thrust acceleration magnitude is chosen to satisfy Eq. (13′), that is, as long as

$$a = W/q \tag{15}$$

When this equation ceases to be satisfied, the intermediate-thrust arc ends. Since a is nonnegative, Eq. (15) implies that W and q must have the same sign. Therefore, neither W nor q can change sign during an intermediate-thrust arc unless the other changes simultaneously; this probability is an unlikely special case. If we also exclude the equally unlikely cases (that are also pathological, since they involve infinite accelerations) of simultaneous vanishment of q and $\dot{q}$, or vanishment of q at an end point of the arc, we have $q \neq 0$ throughout an intermediate-thrust arc. Therefore, such arcs fall into two classes, accordingly, as $q > 0$ and $q < 0$ throughout the arc. It will be shown in the next section that intermediate-thrust arcs with $q < 0$ cannot be optimal. A later section shows that "sufficiently short" arcs with $q > 0$ are optimal. Equation (11′) implies that $(\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})^2 U$ is nonnegative. For an inverse-square central field this gives

$$\sin^2\psi \leq \tfrac{1}{3} \tag{16}$$

where ψ is the angle between $\boldsymbol{\lambda}$ and the local horizontal, so $\boldsymbol{\lambda}\cdot\mathbf{r} = \lambda r \sin\psi$. In the same field, it is easy to show that if μ denotes the gravitational constant, then

$$q = -(3\mu/r^4)[3 - 5\sin^2\psi]\sin\psi \tag{17}$$

Because of Eq. (16), the bracketed quantity is positive. Therefore, q is positive or negative, accordingly, as $\boldsymbol{\lambda}$ points inward toward, or outward from, the center of attraction. The optimality condition $q > 0$ therefore implies an inward-pointing $\boldsymbol{\lambda}$ in the case of an inverse-square field.

3. Derivation of the Necessary Condition $q > 0$

Perturbing equation (1) gives, to second order in δr,

$$\delta\mathbf{a} = \delta\ddot{\mathbf{r}} - (\delta\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{g} - \tfrac{1}{2}(\delta\mathbf{r}\cdot\boldsymbol{\nabla})^2\mathbf{g} \tag{18}$$

The unperturbed trajectory is assumed to be an intermediate-thrust arc, so $\boldsymbol{\lambda}$ has constant magnitude. Since only one such arc is under consideration, there is no loss of generality in choosing this constant magnitude to be unity. Throughout this section and the next, $\lambda \equiv 1$ will be assumed for convenience. Introducing the scalar J and the dyadics $\mathbf{G}$ and $\mathbf{Q}$ defined by

$$J = \delta a - \boldsymbol{\lambda}\cdot\delta\mathbf{a} \tag{19}$$

$$\mathbf{G} = \boldsymbol{\nabla}\mathbf{g} = -\boldsymbol{\nabla}\boldsymbol{\nabla}U = \text{gravity gradient dyadic} \tag{20}$$

$$\mathbf{Q} = -(\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})\mathbf{G} = (\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\boldsymbol{\nabla}U \tag{21}$$

Eq. (3) becomes

$$\ddot{\boldsymbol{\lambda}} = \mathbf{G}\cdot\boldsymbol{\lambda} \tag{22}$$

and Eq. (18) gives

$$\delta a = J + \boldsymbol{\lambda}\cdot(\delta\ddot{\mathbf{r}} - \mathbf{G}\cdot\delta\mathbf{r}) + \tfrac{1}{2}\delta\mathbf{r}\cdot\mathbf{Q}\cdot\delta\mathbf{r} \tag{23}$$

The characteristic velocity P is defined by

$$P = \int a\,dt \tag{24}$$

As is well known [and readily verified by Eqs. (4) and (5)], P/c is the logarithm of the mass ratio, so optimal trajectories must minimize P. By integrating both sides of Eq. (23) over an interval, the variation of P is found to be

$$\delta P = \int(J + \tfrac{1}{2}\delta\mathbf{r}\cdot\mathbf{Q}\cdot\delta\mathbf{r})dt \tag{25}$$

The second term on the right side of Eq. (23) disappears from the result because the perturbations $\delta\mathbf{r}$ and $\delta\dot{\mathbf{r}}$ are assumed to vanish at both ends of the interval; therefore integrating by parts twice and using Eq. (22) gives

$$\int\boldsymbol{\lambda}\cdot(\delta\ddot{\mathbf{r}} - \mathbf{G}\cdot\delta\mathbf{r})dt = 0 \tag{26}$$

The next step is to find a convenient expression for the value of J. Define $\delta a_{\perp}$ to be the component of $\delta\mathbf{a}$ perpendicular to $\boldsymbol{\lambda}$, and let a_0 denote the unperturbed value of a. Then by Eq. (2) and the assumption $\lambda \equiv 1$, the unperturbed value of $\mathbf{a}$ must be $\mathbf{a}_0 \equiv \boldsymbol{\lambda}a_0$, so $\boldsymbol{\lambda}\cdot\mathbf{a}_0 \equiv a_0$. Therefore, Eq. (19) may be written as

$$\begin{aligned} J &= -\boldsymbol{\lambda}\cdot\mathbf{a} + a \\ &= -(a_0 + \boldsymbol{\lambda}\cdot\delta\mathbf{a}) + [(a_0 + \boldsymbol{\lambda}\cdot\delta\mathbf{a})^2 + (\delta a_{\perp})^2]^{1/2} \end{aligned} \tag{27}$$

from which

$$J \geq -(a_0 + \boldsymbol{\lambda}\cdot\delta\mathbf{a}) + |a_0 + \boldsymbol{\lambda}\cdot\delta\mathbf{a}| \tag{28}$$

with equality holding only if $\delta a_{\perp} = 0$ and $a_0 + \boldsymbol{\lambda}\cdot\delta\mathbf{a} \geq 0$. Equations (27) and (28) are valid for strong variations, i.e., variations with no restrictions on the size of $\delta\mathbf{a}$. For the special case of weak variations (small $\delta\mathbf{a}$), Eq. (27) gives

$$J \approx (\tfrac{1}{2}a_0)(\delta a_{\perp})^2 \tag{29}$$

or, to second-order approximation

$$J \approx (\tfrac{1}{2}a_0)|(\delta\ddot{\mathbf{r}} - \mathbf{G}\delta\mathbf{r})_{\perp}|^2 \tag{30}$$

where the subscript $\perp$ still means "perpendicular to $\boldsymbol{\lambda}$." Substituting this into Eq. (25) gives δP in terms of $\delta\mathbf{r}$ and $\delta\ddot{\mathbf{r}}$. To derive the new necessary condition, it is convenient to choose

$$\delta\mathbf{r} = \boldsymbol{\lambda}\ddot{u} - 2\dot{\boldsymbol{\lambda}}\dot{u} + 4\ddot{\boldsymbol{\lambda}}u \tag{31}$$

where the function $u(t)$ and its first three derivatives are continuous, and the fourth derivative of u is continuous except for a finite number of jump discontinuities. Also

$$0 = u = \dot{u} = \ddot{u} = (\dddot{u}) \tag{32}$$

everywhere outside a subinterval in which $q = \boldsymbol{\lambda}\cdot Q\cdot\boldsymbol{\lambda}$ is negative. It is easy to verify that

$$\delta\ddot{\mathbf{r}} - \mathbf{G}\cdot\delta\mathbf{r} = \boldsymbol{\lambda}u^{(4)} + (6\dddot{\boldsymbol{\lambda}} + 2\mathbf{G}\cdot\dot{\boldsymbol{\lambda}})\dot{u} + 4(\boldsymbol{\lambda}^{(4)} - \mathbf{G}\cdot\ddot{\boldsymbol{\lambda}})u \tag{33}$$

so the component of $\delta\ddot{\mathbf{r}} - \mathbf{G}\cdot\delta\mathbf{r}$ normal to $\boldsymbol{\lambda}$ involves only u and $\dot{u}$. In consequence, if $u(t)$ is chosen to be rapidly oscillatory at some high frequency ω, the integral of J is of order ω^2. But since $\delta\mathbf{r}$ involves $\ddot{u}$, the integral of $\delta\mathbf{r}\cdot\mathbf{Q}\cdot\delta\mathbf{r}$ is of order ω^4, and dominates if ω is chosen sufficiently large. That is, for sufficiently large ω we can write

$$\delta P \approx \tfrac{1}{2}\int(\ddot{u})^2\boldsymbol{\lambda}\cdot\mathbf{Q}\cdot\boldsymbol{\lambda}\,dt \tag{34}$$

These statements can be readily proved by standard methods if a specific form is chosen for $u(t)$. A convenient choice for u, inside the subinterval, is

$$u(t) = [(t_2 - t)(t - t_1)]^3 \sin(\omega t) \tag{35}$$

where t_1 and t_2 are the two ends of the subinterval. Since by hypothesis $q = \boldsymbol{\lambda}\cdot\mathbf{Q}\cdot\boldsymbol{\lambda}$ is negative throughout the interval, Eq. (34) shows δP to be negative, indicating nonoptimality of the original trajectory.

We have therefore proven that $q \geq 0$ is a necessary condition for optimality of an intermediate-thrust arc. But it was pointed out in Sec. 2 that intermediate-thrust arcs have $q \neq 0$ throughout, except in some unlikely special cases. Therefore, the necessary condition may as well be stated in the stronger form $q > 0$. For inverse-square fields, this means that the primer vector must have a negative radial component, i.e., must point inward.

The necessary condition just proved has a very simple intuitive interpretation. If the thrust magnitude is modulated at a sufficiently high frequency, without altering its average value, the principal effect is to cause oscillatory position deviations in a direction parallel to $\boldsymbol{\lambda}$. These in turn cause additional accelerations due to the position dependence of $\mathbf{g}$. The contribution from gravity gradient is of alternating sign and averages out, but the second derivative of gravity with respect to position gives a contribution of constant sign, and of second order in the perturbation parameter. Its average value is approximately

$$\tfrac{1}{2}(|\delta r|^2{}_{\text{av}})(\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})^2\mathbf{g} \tag{36}$$

To preserve the average motion, this extra acceleration must be compensated by an equal and opposite change in the average thrust acceleration $\mathbf{a}$. The corresponding change in average thrust acceleration magnitude is

$$-\tfrac{1}{2}(|\delta\mathbf{r}|^2{}_{\text{av}})(\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})^2\boldsymbol{\lambda}\cdot g = \tfrac{1}{2}q|\delta\mathbf{r}|^2{}_{\text{av}} \tag{37}$$

so if $q < 0$, the trajectory is "improved" by an oscillatory perturbation, and therefore cannot be optimal. It is interesting to note that although an arbitrarily high frequency ω was assumed for the purpose of proving the necessary condition, the greatest "improvement" occurs at a finite frequency. This is because the amplitude of acceleration modulation is limited by constraints independent of frequency (i.e., the requirements that $0 \leq a \leq F/m$ and that $\delta a_\perp$ be "small") so the greatest mean-square position deviation, and hence the greatest effect on total impulse, is obtained by reducing the frequency to some "best" finite value. Below this "best" value, effects which were relatively negligible at high frequencies become more important, and the impulse saving becomes smaller or disappears.

The intuitive argument just given can be readily made rigorous and generalized to other singular optimal-control problems. The basic idea is to consider a control variation consisting of an oscillatory variation of the singular control during a very short time interval, plus a higher-order variation of all control variables over the same or a more extended interval. The latter variation serves to preserve the final conditions, provided the system has the required controllability (normality) property. This is essentially the approach used by Kelley[4] to derive a generalization of the Legendre-Clebsch condition that applies to the singular case of optimal control. Kelley's necessary condition (that involves the partial derivative, with respect to the singular control variable, of the second time derivative of the switching function K) is trivially satisfied for intermediate-thrust trajectories in vacuum, and hence gives no information as to their optimality. The necessary condition derived in the present paper involves the partial of the fourth time derivative of the switching function, and is evidently analogous to Kelley's condition but applies to a "more singular" case. It can be derived by a straightforward extension of Kelley's methods, as has been shown by Kopp and Moyer[5] of the Grumman Aircraft Engineering Corporation.

4. Sufficiency Proof

This section will show that, if an intermediate-thrust arc satisfies the necessary conditions given in Sec. 2 and also satisfies the condition $q \geq q_0$ where q_0 is some positive constant, then any "sufficiently short" segment of this arc is an absolute optimum solution to the trajectory optimization problem defined by taking the end points of the segment as initial and final conditions. That is, it is an absolute optimum with respect to variations of the interior of the segment, keeping position and velocity fixed at the segment's end points. The phrase "sufficiently short" means that the duration Δt of the segment is no greater than a certain bound that can be estimated. No effort was made to find the best possible bound, since the only purpose of the argument is to show that there are at least *some* trajectory problems for which the intermediate-thrust arc gives the absolute optimum, and to show that no additional *local* necessary conditions remain to be found. (There is presumably an additional *nonlocal* necessary condition, which is a generalization of Jacobi's condition, and which states that the optimal arc must not contain a pair of conjugate points.)

If a_u is an upper bound for $a_0(t)$ on the intermediate-thrust arc, then the total impulse of the unperturbed segment is $\leq a_u\Delta t$. The same bound must apply for any "better" perturbed segment between the same end points. Therefore, we need only consider perturbations that satisfy

$$\int|\delta\mathbf{a}(t)|\,dt \leq 2a_u\Delta t \tag{38}$$

where the integral is over the segment. This relation follows from the triangle inequality $|\mathbf{a} - \mathbf{a}_0| \leq a + a_0$, together with the bounds just stated. If c_1 is any constant greater than $2a_u$, Eq. (38) implies the inequalities

$$|\delta\dot{\mathbf{r}}(t)| \leq c_1\Delta t \tag{39}$$

$$|\delta\mathbf{r}(t)| \leq c_1(\Delta t)^2 \tag{40}$$

for Δt sufficiently small. Therefore, if Δt is sufficiently small, $|\delta\mathbf{r}|$ is guaranteed to be small enough to justify the neglect of terms of order $|\delta\mathbf{r}|^3$ or higher in Eq. (25), and absolute optimality is proven if the integral in Eq. (25) can be proved to be always nonnegative and to be zero only if $\delta\mathbf{r}$ is identically zero. The first step is to note that because of the hypothesis $q > q_0$, it is obviously possible to find a positive number s_0 such that

$$\delta\mathbf{r}\cdot\mathbf{Q}\cdot\delta\mathbf{r} \geq q_0(\delta r_\lambda)^2 - s_0(\delta r_\perp)^2 \tag{41}$$

where δr_λ and $\delta r_\perp$ are the components of $\delta\mathbf{r}$ parallel to and normal to $\boldsymbol{\lambda}$, respectively. Then, defining δP_0 by the equation

$$2\delta P_0 = \int[q_0(\delta r_\lambda)^2 - s_0(\delta r_\perp)^2]dt \tag{42}$$

optimality will be proved if it can be shown that

$$0 \leq \delta P_0 + \int J\,dt \tag{43}$$

where J is defined by Eq. (19 or 27).

The next step is to find a convenient lower bound for the J integral. Introducing the abbreviations

$$A = \int a\,dt \tag{44}$$

$$B = \int a_\perp dt \tag{45}$$

it is a trivial problem in the calculus of variations to show that

$$\int J\,dt \geq A - (A^2 - B^2)^{1/2} \geq B^2/A \tag{46}$$

But it has already been shown that we need only consider trajectories for which $A \leq a_u\Delta t$. For such trajectories we have

$$\int J\,dt \geq B^2/(a_u\Delta t) \tag{47}$$

From Eqs. (42, 43, and 47), it is clear that the desired result will be proved if it can be shown that

$$s_0\int(\delta r_\perp)^2dt \leq q_0\int(\delta r_\lambda)^2dt + 2B^2/(a_u\Delta t) \quad (48)$$

To get an upper bound for the integral of $(\delta r_\perp)^2$, it is convenient to introduce a rotating-coordinate system that has its third axis always parallel to $\boldsymbol{\lambda}$ and has zero angular rate about this axis. In this coordinate system, the equations of perturbed motion become:

$$\delta\dot{r}_1 = \delta v_1 + \omega_2\,\delta r_3 \quad (49a)$$

$$\delta\dot{r}_2 = \delta v_2 - \omega_1\,\delta r_3 \quad (49b)$$

$$\delta\dot{r}_3 = \delta v_3 - \omega_2\,\delta r_1 + \omega_1\,\delta r_2 \quad (49c)$$

$$\delta\dot{v}_1 = \delta a_1 + \omega_2\,\delta v_3 + (\mathbf{G}\cdot\delta\mathbf{r})_1 \quad (50a)$$

$$\delta\dot{v}_2 = \delta a_2 - \omega_1\,\delta v_3 + (\mathbf{G}\cdot\delta\mathbf{r})_2 \quad (50b)$$

$$\delta\dot{v}_3 = \delta a_3 - \omega_2\,\delta v_1 + \omega_1\delta v_2 + (\mathbf{G}\cdot\delta\mathbf{r})_3 \quad (50c)$$

where ω_1 and ω_2 are components of $\boldsymbol{\omega}$, the angular velocity of the coordinate system. Equations (49a) and (49b) may be combined to give

$$|(\delta r_\perp)^\cdot| \leq \delta v_\perp + \omega|\delta r_3| \quad (51)$$

Using Eq. (49c) to eliminate δv_3 from Eqs. (50a) and (50b) gives the new equations

$$(\delta v_1 - \omega_2\delta r_3)^\cdot = \delta a_1 + K_1 \quad (52a)$$

$$(\delta v_2 + \omega_1\delta r_3)^\cdot = \delta a_2 + K_2 \quad (52b)$$

where

$$K_1 = (\mathbf{G}\cdot\delta\mathbf{r})_1 - \dot{\omega}_2\delta r_3 + \omega_2(\omega_2\,\delta r_1 - \omega_1\,\delta r_2) \quad (53a)$$

$$K_2 = (\mathbf{G}\cdot\delta\mathbf{r})_2 + \dot{\omega}_1\delta r_3 - \omega_1(\omega_2\,\delta r_1 - \omega_1\,\delta r_2) \quad (53b)$$

No use will be made of Eq. (50c). If c_2 is an upper bound for ω, Eq. (51) implies

$$\delta r_\perp \leq \int\delta v_\perp\,dt + c_2\int|\delta r_3|\,dt \quad (54)$$

Similarly, Eqs. (52a) and (52b) imply

$$\delta v_\perp \leq c_2|\delta r_3| + \int a_\perp dt + \int K_\perp dt \quad (55)$$

where $K_\perp$ is the square root of $(K_1)^2 + (K_2)^2$. Since $\dot{\omega}_1$, $\dot{\omega}_2$, and the components of $\mathbf{G}$ are bounded, there must exist a constant c_3 such that

$$K_\perp \leq c_3\,\delta r \quad (56)$$

where $\delta r = |\delta\mathbf{r}|$. Applying Schwarz's inequality to the last term of Eq. (55), and recalling the definition of B, gives

$$\delta v_\perp \leq c_2|\delta r_3| + B + c_3[\Delta t\int(\delta r)^2dt]^{1/2} \quad (57)$$

Integrating again, and substituting into Eq. (54), gives

$$\delta r_\perp \leq B\Delta t + 2c_2\int|\delta r_3|\,dt + c_3\Delta t[\Delta t\int(\delta r)^2dt]^{1/2} \quad (58)$$

Or, using $|\delta r_3| \leq \delta r$ and applying Schwarz's inequality again

$$\delta r_\perp \leq B\Delta t + (2c_2 + c_3\Delta t)[\Delta t\int(\delta r)^2dt]^{1/2} \quad (59)$$

This equation can be converted to a more convenient form by use of the general inequality $(x + y)^2 \leq 2x^2 + 2y^2$. The result is

$$(\delta r_\perp)^2 \leq 2(B\Delta t)^2 + 2(2c_2 + c_3\Delta t)^2\Delta t\int(\delta r)^2dt \quad (60)$$

Integrating over the segment gives the final result

$$\int(\delta r_\perp)^2dt \leq 2B^2(\Delta t)^3 + 2(2c_2 + c_3\Delta t)^2(\Delta t)^2\int(\delta r)^2dt \quad (61)$$

Compare this with Eq. (48), rewritten in the form

$$(s_0 + q_0)\int(\delta r_\perp)^2dt \leq q_0\int(\delta r)^2dt + 2B^2/(a_u\Delta t) \quad (62)$$

It is evident that Eq. (61) implies Eq. (62) (and therefore guarantees optimality) if Δt is chosen sufficiently small to satisfy the two conditions

$$(s_0 + q_0)a_u(\Delta t)^4 < 1 \quad (63)$$

and

$$2(s_0 + q_0)(2c_2 + c_3\Delta t)^2(\Delta t)^2 < q_0 \quad (64)$$

Furthermore, Eqs. (62) and (48) hold in their strong form (strict inequality) unless $\delta r \equiv 0$. This completes the sufficiency proof.

5. Junction Conditions

This section will review the conditions for joining intermediate-thrust arcs to other types of arcs in an optimal trajectory. The condition $q > 0$ puts these conditions in a new light, reducing the number of cases that need be considered.

First, let us consider the junction of an intermediate-thrust arc to an arc of finite maximum thrust, assuming $a < a_{\max}$ on the intermediate-thrust side of the junction. Since $a = a_{\max}$ on the other side of the junction, $a(t)$ has a jump discontinuity at the junction. By Eq. (13), this implies a jump discontinuity in $\lambda^{(4)}$ at the junction. Assuming $q > 0$, $\lambda^{(4)}$ is negative on the maximum-thrust side of the junction, which makes $\lambda(t)$ *decrease* away from the junction on the maximum-thrust side. But this is incompatible with the necessary conditions given in Sec. 2, which require $\lambda(t)$ to *increase* on going from an intermediate-thrust arc to a maximum-thrust arc. Therefore, an intermediate-thrust that satisfies the condition $q > 0$ cannot be joined to a maximum-thrust arc unless $a = a_{\max}$ on both sides of the junction, which represents a forced transition (further extrapolation of the intermediate-thrust arc would violate the condition $a \leq a_{\max}$) and is an improbable special case.

An analogous discussion can be given for the junction of an intermediate-thrust arc directly (i.e., without an intervening impulse) to a coast arc. If $a > 0$ on the intermediate-thrust side of the junction, discontinuities of a and of $\lambda^{(4)}$ occur at the junction point. For positive q, $\lambda^{(4)}$ is positive on the zero-thrust side of the junction, so λ (t) *increases* away from the junction on the zero-thrust side. But, this is incompatible with the necessary conditions given in Sec. 2, which call for a *decrease* in λ (t) on going from an intermediate-thrust arc to a zero-thrust arc. Therefore, an intermediate-thrust arc with $q > 0$ cannot join directly to a zero-thrust arc unless $a = 0$ on both sides of the junction, an unlikely special case that corresponds to a transition *forced* by the requirement $a \geq 0$.

If there is no limit to thrust magnitude (i.e., if impulses are allowed) the situation changes. Equations (12) and (13) show that an impulse causes a step discontinuity in $\dddot{\lambda}$. If $q > 0$, the discontinuity is a step decrease in $\dddot{\lambda}$. Therefore, if impulses occur at both ends of an intermediate-thrust arc that is preceded and followed by coast arcs, $\dddot{\lambda}$ changes from some positive value to zero at the first impulse, and from zero to some negative value at the second impulse. In each case, $\lambda(t)$ *decreases* away from the junction on the coast-arc side. This is compatible with the necessary conditions given in Sec. 2. Therefore, an intermediate-thrust arc can join to a zero-thrust arc if a nonzero impulse occurs at the junction.

The extremal trajectories that begin at a given initial state at time t_0 and include an intermediate-thrust arc with impulses at both ends constitute a six-parameter family. To see this, note than an extremal in this class can be generated by choosing the eleven variables, $\boldsymbol{\lambda}_0$ = initial primer vector (three variables), $\dot{\boldsymbol{\lambda}}_0$ = initial derivative of $\boldsymbol{\lambda}$ (three variables), K_0 = initial value of $K(t)$, t_b = beginning of intermediate-thrust arc, t_e = end of intermediate-thrust arc, and P_b, P_e = impulses at t_b, t_e to satisfy the four necessary conditions $K_b = \dot{\lambda}_b = \ddot{\lambda}_b = \dddot{\lambda}_b = 0$ and the scaling convention $\lambda_b = 1$. This six-parameter family gives just enough freedom to satisfy the six final conditions given by a specification of position and velocity at the final time t_f. In the finite-thrust case (with $q > 0$ required on the intermediate-thrust arc) there are two less variables (P_b, P_e are no longer available) and two more constraints since $a(t)$ must equal zero or $a_{\max}$ at each of the times t_b, t_e. Therefore, in the finite-thrust case, an extremal including an intermediate-thrust arc and satisfying all necessary conditions cannot satisfy general initial and final conditions. Rather, four relations must hold among the initial and final conditions to make possible an optimal finite-thrust solution with intermediate-thrust arc. Therefore, such solutions only occur in a set of cases of measure zero. For the impulsive case, however, optimal solutions with intermediate-thrust arcs occupy some finite volume in the space of boundary condition parameters.

6. Conclusions

Lawden[2, 3] has published equations by which one can find all intermediate-thrust arcs in two dimensions for an inverse-square field. In Lawden's notation, the new necessary condition is simply

$$\sin\psi < 0 \tag{65}$$

but its application is complicated by the need to verify the fact that ψ has been chosen in a way that makes the quantity f (Lawden's symbol for the magnitude of thrust acceleration) positive as it should be. Examination of Lawden's solutions shows that some of them satisfy the new condition, and therefore give absolute optima for at least some sets of boundary conditions, whereas others violate the new condition and must be nonoptimal. In particular, Lawden's spiral with $C = 0$ is nonoptimal.

Although optimal intermediate-thrust arcs exist, they seem to be without practical significance because of the restrictive junction conditions. It has been shown that if any limit, however large, is imposed on the thrust magnitude, then the optimal solution to a trajectory problem defined by given boundary conditions cannot include an intermediate-thrust arc except in a set of special cases which is of measure zero. Yet, if no thrust limit is imposed (i.e., if impulses are allowed) optimal trajectories with intermediate-thrust arcs occupy a nonzero volume in the boundary-condition space. The obvious implication is that inside this volume, the optimal finite-thrust trajectories alternate between zero and maximum thrust at a frequency that goes to infinity as the limit of allowed thrust level is raised to infinity. The probable dependence is $\omega^4 \sim a_{\max}$ with a constant of proportionality that depends on the boundary conditions and goes to infinity as these conditions approach any of the special cases for which an optimal finite-thrust trajectory can have an intermediate-thrust arc. This conclusion is of theoretical rather than practical interest since the fuel penalty incurred by limiting the burn periods to a small number (i.e., ≤ 4) will generally be minute. In general, the results of this paper support the prevailing belief that intermediate-thrust arcs have only theoretical significance.

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